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A Dynamic Programming Analysis of Multiple Guidance Corrections of a Trajectory

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ABSTRACT

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The problem of deciding when to apply guidance corrections to the perturbed trajectory of a spacecraft is treated from the dynamic programming point of view. It is assumed that the objective of the guidance correction policy is to minimize the expected value of the squared error at the final time, subject to the constraint that the total correction capability expended be less than some specified value. It is shown that a correction should be performed when a certain "switching function" passes through zero. Assuming that the orbit determination procedure has been prespecified, and that the statistics of the correction errors are known, the switching function is found to depend upon the instantaneous state of the system, which is composed of (1) the estimate of the trajectory perturbation to be corrected, (2) the variance of the error in this estimate, and (3) the correction capability of the spacecraft. Equations for computing the switching function are derived, and a numerical example is presented.

AUTHOR

I. INTRODUCTION

A spacecraft traversing a coast trajectory toward some target region in space is guided to its final destination by applying one or more small velocity impulse corrections (maneuvers) at certain times along the path so as to null the predicted target error. The prediction (estimate) of the target error is achieved by an orbit determination process; the required corrections are computed using linear perturbation theory, and the impulse is delivered by a rocket motor, which applies an acceleration to the spacecraft for a relatively short period of time. The selection of times for performing the velocity corrections to the orbit, and the determination of what fraction of the predicted target error is to be nulled by each correction, is termed the *guidance policy*. It is the purpose of this report to develop a guidance policy that will effectively maximize the probability of impacting the target region, subject to the constraint that the total propellant expended in performing the corrections is less than some prespecified amount.

Defining the guidance policy is an easy task if the orbit is perfectly known, if the correction can be made perfectly, and if there is adequate correction capability (propellant). Otherwise the policy is not readily constructed. There are factors that tend to cause a maneuver to be made early, such as the smaller amount of correction capability required to null a given target error; and there are factors that tend to cause it to be made late, such as the need to process more data to get a better estimate of the orbit. The random errors arising in the execution of the correction must be considered, since they affect the uncertainty in the knowledge of the orbit parameters. The problem, then, is to develop a guidance policy that will allot the given correction capability in such a way as to cause some penalty function to be minimized, taking into account the uncertainty arising from orbit determination and execution errors.

The theory concerning the single-impulse correction is well known (Ref. 1) and was implemented in the successful Mariner II fly-by mission to the planet Venus (Ref. 2). In this case, a suitable single maneuver time is chosen from preflight studies of orbit determination and execution error statistics, and the correction capability to be carried aboard the spacecraft is determined by mapping the covariance matrix of injection guidance errors to the selected maneuver point to obtain the covariance matrix of velocity-to-be-gained components. The situation becomes much more complex when more than one maneuver is considered, for then the future guidance and tracking policy must be considered in performing a correction at any given time. It becomes necessary, in general, to consider both the present and future uncertainty in the knowledge of the orbit *and* the errors in the measurement devices being used to determine the orbit. The target error criterion and desired accuracy must be defined, as well as the bound on the total velocity correction that can be applied. This important inquiry has recently received considerable attention by treating it as an optimization problem and has been attacked from several different points of view by Battin, Breakwell, Striebel, and Lawden (Ref. 3 through 6, respectively). The analysis presented here approaches the problem from the dynamic programming point of view (Ref. 7), defining an optimal policy as one which minimizes the mean squared target error, subject to constraints on the total correction capability that can be allotted. This scheme considers the time-varying estimate of the trajectory perturbation to be corrected and the variance of the error in this estimate, leading to a guidance policy that is trajectory dependent.

The nomenclature used is as follows: A bold face letter represents a column vector; an asterisk indicates an estimated quantity; the symbol E [—] indicates the statistical expectation (average value) over all similar experiments of the quantity in brackets; matrices are denoted by capital letters; and the superscript T indicates a matrix or vector transpose. The word *uncertainty* will be used synonymously with the word *variance*.

II. SUMMARY

An idealized guidance problem is defined, assuming that a series of velocity impulse corrections are to be applied to the trajectory of the spacecraft while it is traveling in a straight line toward impact on a massless planet. The equations describing the orbit determination and guidance policy is prespecified, i.e., the types of observed data to be gathered throughout the entire mission, and the times for making these observations, are known from preflight studies and do not depend upon the guidance policy. The statistics of the errors

The penalty function p to be minimized is defined as the sum of the orbit determination uncertainty immediately after the final correction (at the prespecified final time t_f) plus the square of the uncorrectable error due to depleting the correction capability at t_f, i.e.,

$$p \stackrel{\Delta}{=} \beta_f + r^2 \tag{1}$$

where β_f is the final orbit determination variance, and r is the estimate of the target error immediately after the correction at t_f. The case $r \neq 0$ occurs when there is not sufficient correction capability at t_f, and a full correction of the estimated error cannot be made. It is shown that minimizing the penalty function p can be interpreted as being equivalent to maximizing the probability of impacting the target planet.

A sequence of "decision times" t_i are defined along the trajectory, where the possibility of performing a correction is to be examined. At each time $t_i < t_f$ the *expected value* of p, as a function of the instantaneous state of the system, is substituted for p in the optimization problem. The state of the system x at any time t_i is considered to be composed of

1. The minimum variance estimate of the uncorrected target error m_i^*, which is obtained from the orbit determination process by considering all data (including the *a priori* estimate) gathered prior to t_i.

2. The variance of the error in this estimate.

3. The amount of velocity correction capability that can be allotted during the remainder of the mission.

The optimization problem is formulated from the dynamic programming point of view, and it is shown that at each time t_i there should be either a total correction of the estimated error, or no correction at all. From this conclusion the optimal guidance policy is implemented as follows:

1. The expected value of p is calculated at t_i, assuming a total correction at t_i and t_f; i.e.,

$$\mathrm{E}\left[p(x)\right]_i = \mathrm{E}\left[\beta_f(x)\right]_i + \mathrm{E}\left[r^2(x)\right]_i \tag{2}$$

2. The quantity $\mathrm{E}\left[p_0(x)\right]_i$ is calculated at t_i, which is the expected value of p, assuming no corrections except at the final time t_f.

3. If $\mathrm{E}\left[p(x)\right]_i - \mathrm{E}\left[p_0(x)\right]_i \geq O$ make no correction at t_i; go on to the next decision-time t_{i+1}.

4. If the above inequality does not hold, the quantity $\mathrm{E}\left[p_{i+1}(x)\right]_i$ is calculated at t_i, which is the expected value of p, assuming a total correction at t_{i+1} and t_f, but none at t_i. This computation is made possible at t_i by recognizing that the expected value of the estimate of the target error m^* at t_{i+1} is the current estimate, i.e.

$$\mathrm{E}\left[m_{i+1}^*\right]_i = m_i^* \tag{3}$$

5. The switching function defined by

$$s_i = \mathrm{E}\left[p(x)\right]_i - \mathrm{E}\left[p_{i+1}(x)\right]_i \tag{4}$$

is formed. If s_i is positive no action is taken; if it is negative or zero a full correction is applied at t_i.

6. When the next decision-time is reached the process is reinitiated, this time with a new estimate of the error m_{i+1}^*, based upon the action taken at t_i and the tracking data received during the interval.

The case of insufficient correction capability to accomplish the mission and the case of a limited number of corrections are discussed. Numerical results are presented. The extension to the more general case is discussed.

III. DESCRIPTION OF THE IDEALIZED GUIDANCE PROBLEM

The essential ideas of this report are developed by considering the idealized one-dimensional problem described below. In part VIII the extension of the problem to the more general case is discussed.

The one-dimensional problem is constructed by imagining that the spacecraft is moving in a zero-gravity field at known speed V toward a massless target, and the time-to-go to closest approach is known. A series of velocity impulse corrections perpendicular to the direction of motion can be accomplished at any or all of the prespecified decision times $(t_0, t_1, \ldots t_f)$, where t_0 is the time of beginning the problem and t_f is the final time. The objective of the guidance system is to impact the center of the planet as closely as possible, i.e., to minimize the squared t[illegible]rror (Fig. 1). If a correction is made at t_i it wi[illegible]

$$v_i = -\left(\frac{d_i m_i^*}{\tau_i}\right) \tag{5}$$

where

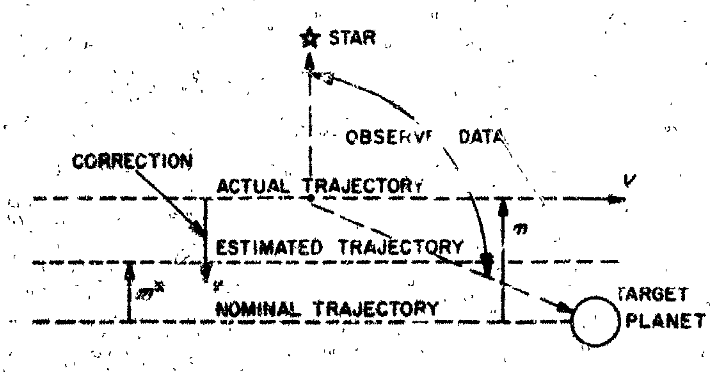
m_i^* is the estimate of the target error at t_i, obtained from the orbit determination process.

τ_i is the time-to-go to closest approach at t_i thus $\tau_i = t_{\text{closest approach}} - t_i$.

d_i is the decision variable, which determines the fraction of the estimate to be nulled at t_i $(0 \leq d_i \leq 1)$.

Between any two decision times (t_i, t_{i+1}) the minimum variance estimate of the target error Δm_i^* is obtained from the orbit determination process in that interval. The variance of the error in that estimate is γ_i. If m_i^* was the previously obtained minimum variance estimate at t_i, with variance α_i, the combined estimate at t_{i+1} is

$$m^*_{i+1} = [\alpha_i^{-1} + \gamma_i^{-1}]^{-1}[\alpha_i^{-1} m_i^* + \gamma_i^{-1}\Delta m_i^*] \tag{6}$$



The variance of the combined estimate is

$$\alpha_{i+1} = [\alpha_i^{-1} + \gamma_i^{-1}]^{-1} = \left[\frac{\alpha_i \gamma_i}{\alpha_i + \gamma_i}\right] \tag{7}$$

At time t_0 the m_i^* and α_i are the *a priori* values.

If a correction is made at t_i there will be further uncertainty added to the knowledge of the target error because of the random execution errors that arise in accomplishing the correction. Thus,

$$\beta_i = \alpha_i + \mathrm{E}[a^2](d_i m_i^*)^2 + \mathrm{E}[b^2]\tau_i^2 \tag{8}$$

where

β_i is the target error variance immediately after the correction at t_i.

$\mathrm{E}[a^2]$ is the variance of the proportional type of execution error (expressed as a decimal fraction).

$\mathrm{E}[b^2]$ is the variance of the nonproportional type of velocity execution error (expressed in m²/sec²).

The assumption will be made that the execution error causes a transverse position displacement without affecting the uncertainty in the direction of the velocity vector, thereby simplifying the subsequent orbit determination process. If a correction is made at t_i the quantity β_i is substituted for α_i in Eq. (6) and (7). If no further corrections are made until the final time t_f, then the uncertainty at t_f, resulting from a correction at t_i, and the orbit determination between t_i and t_f will be

$$\omega_i = \left[\frac{1}{\beta_i} + \frac{1}{\rho_i}\right]^{-1} = \left[\frac{\beta_i \rho_i}{\beta_i + \rho_i}\right] \tag{9}$$

where

$$\rho_i^{-1} = \sum_{j=i}^{f-1} \gamma_j^{-1} \tag{10}$$

The variance of the *estimate* at t_f (as distinguished from the error in the estimate) predicted at t_i, is (Ref. 8).

$$\Psi_i = \mathrm{E}[m_f^{*2}] = \beta_i - \omega_i = \left[\frac{\beta_i^2}{\beta_i + \rho_i}\right] \tag{11}$$

The above equations are employed to determine the guidance policy at each time t_i, which consists of selecting the value of d_i. Thus, for a given orbit determination

tion (correction) error statistics; the policy at t_i is a function of the state of the system at t_i, which will be defined as,

$$x(t_i) = (m^*, \sigma, c)_i \tag{12}$$

where m^* is as defined above, $\sigma = (\alpha)^{1/2}$ and c is the correction capability, expressed in meters/sec. The value c is used in the constraint equation

$$\sum_{j \geq i} v_j \leq c \tag{13}$$

IV. THE PENALTY FUNCTION

Let the specified objective of the guidance policy be to minimize the expected value of the squared error in the closest-approach distance m where $m=0$ on the standard trajectory. Thus, the penalty function to be minimized at t_i is

$$\mathrm{E}\left[p(x)\right]_i = \mathrm{E}\left[\beta_f(x)\right]_i + \mathrm{E}\left[r^2(x)\right]_i \tag{14}$$

where

$\mathrm{E}\left[\beta_f\right]_i$ is the expected value of the uncertainty in m, considering all corrections between times t_i and t_f.

$\mathrm{E}\left[r^2\right]_i$ is the expected value of the residual error in m due to depleting the correction capability between t_i and t_f.

The motivation for choosing this penalty function is that the resultant guidance policy effectively maximizes the probability of achieving a closest-approach distance less than some given limit. This statement is verified below.

Suppose r and ν [$\nu = (\beta_f)^{1/2}$] are, respectively, the mean and standard deviation of the normal distribution of m_f (Fig. 2). Thus,

$$\mathrm{Prob}\,(-\ell \leq m_f \leq \ell) = \int_{-\left(\frac{\ell+r}{\nu}\right)}^{+\left(\frac{\ell-r}{\nu}\right)} f(z)\,dz \tag{15}$$

where ℓ is a given limit, and

$$f(z) = \frac{1}{(2\pi)^{1/2}} \exp - \left(\frac{z^2}{2}\right) \tag{16}$$

If r is assumed small, Eq. (15) may be written[1]

$$\mathrm{Prob.}\,(-\ell \leq m_f \leq \ell) \approx \int_{-\left(\frac{\ell}{\nu}\right)}^{+\left(\frac{\ell}{\nu}\right)} f(z)\,dz - \left[\left(\frac{\ell}{\nu}\right)\left(\frac{r}{\nu}\right)^2\right] f\left(\frac{\ell}{\nu}\right)$$

$$\approx \int_{-\frac{\ell}{(p)^{1/2}}}^{+\frac{\ell}{(p)^{1/2}}} f(z)\,dz \tag{17}$$

where

$$p = \nu^2 + r^2 = \beta_f + r^2 \tag{18}$$

For any given value of ℓ expression (17) is clearly maximized by minimizing p. Since only the expected value of p can be computed at t_i, the penalty function given by Eq. (14) is a reasonable one.

Anticipating the analysis to follow, suppose a total correction ($d_i = 1$) is made at t_i, and consider the evaluation of the expected value of p. Since m^* has been

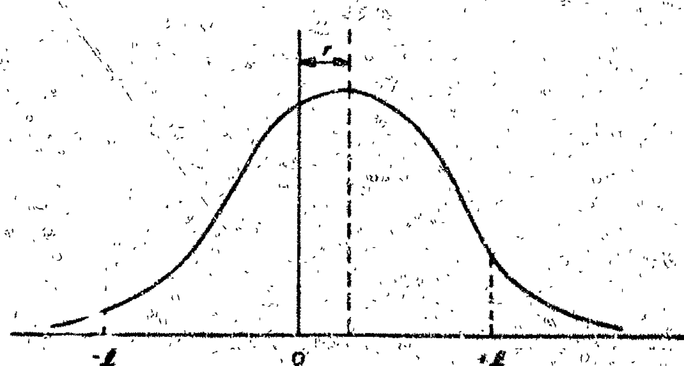


Fig. 2. The biased probability density function

[1] This "equivalence" of bias and standard deviation was pointed out in an unpublished paper by T. W. Hamilton of the Jet Propulsion Laboratory.

lled at t_i, it follows that the predicted value of m^* is zo for all future times, and no further corrections are pected until t_f. The correction capability remaining to applied at the final time t_f is

$$c_f = c(t_f) = c(t_i) - \left(\frac{m_i^*}{\tau_i}\right) \tag{19}$$

y estimate $m_f^* \leq c_f \tau_f$ can be nulled at t_f, resulting in m_f^*), as shown in Fig. 3. Thus, the expected value of r^2, aluated at t_i, is

$$\mathrm{E}\left[r^2\right]_i = 2\Psi_i \int_{\lambda}^{\infty} f(z)(z-\lambda)^2\, dz \tag{20}$$

ere

$$\lambda_i = \frac{c_f \tau_f}{(\Psi_i)^{1/2}} \tag{21}$$

Ψ_i is defined by Eq. (11). Thus, the expected value , evaluated at t_i by assuming a total correction at t_i t_f, is

$$\left[p(x)\right]_i = \Psi_i\, g(\lambda_i) + \omega_i + \Psi_i\, \mathrm{E}\left[a^2\right] + r_f^2\, \mathrm{E}\left[b^2\right] \tag{22}$$

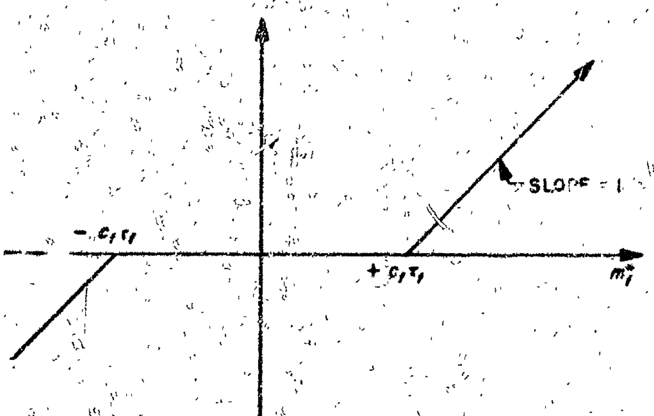


Fig. 3. The residual error at the final time

where the "residual function" is (Fig. 4)

$$g(\lambda) = 2 \int_{\lambda}^{\infty} f(z)(z-\lambda)^2\, dz \tag{23}$$

and ω_i is defined by Eq. . It will be shown in part VI that it is indeed correct to assume that a total correction should be made whenever a correction is called for, and that $E\left[p(x)\right]_i$ can therefore be evaluated from Eq. 22.

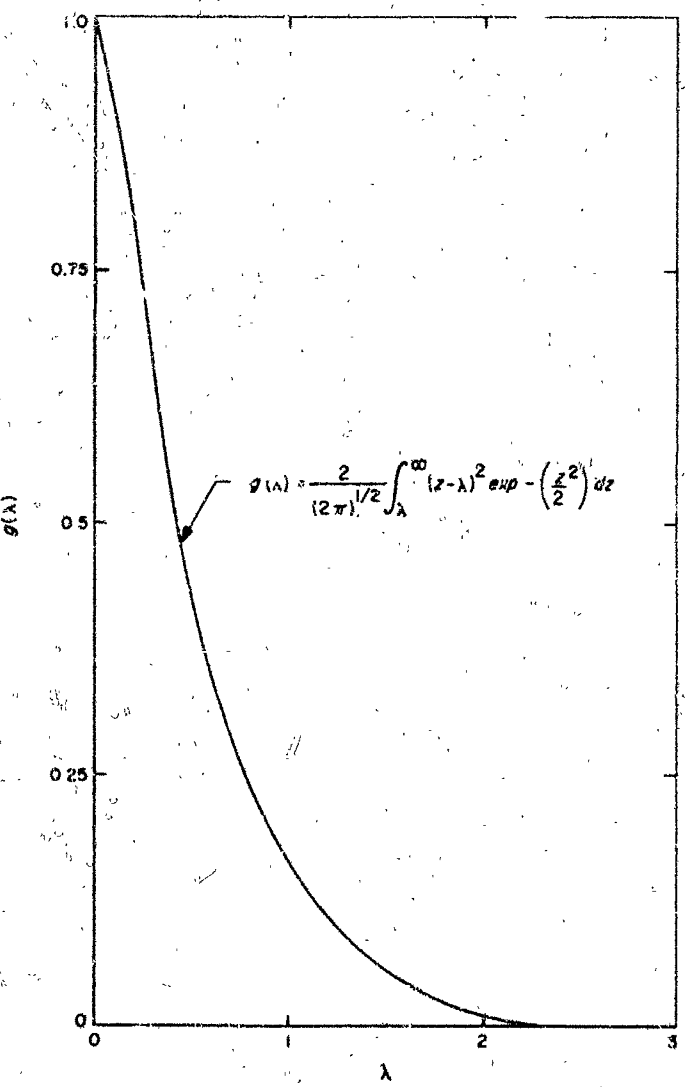


Fig. 4. The residual function

V. THE DYNAMIC PROGRAMMING FORMULATION

The guidance policy, which minimizes the penalty function discussed in the previous part, can be formulated by invoking the principle of optimality of dynamic programming (Ref. 7), which states: *An optimal policy has the property that whatever the initial state and initial decision is, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

This principle is applied here by imagining a set of tables at each time t_i which presents the minimum expected value of the penalty function p and the associated decision variable d as a function of the state-variables of the system, which are the predicted target error m^*, the standard deviation of the error in this estimate σ and the correction capability c. These tables would be constructed by working backwards from the final time, at each t_i considering all conceivable combinations of state-variables. At each t_{i-1} the decision and penalty are arrived at by finding the decision that will transfer the state to the subsequent decision time t_i in such a way as to obtain minimum penalty, which is evaluated by interpolating the state-variables in the previously computed table at t_i. The mathematical formulation is as follows: Let

d_i = the decision at t_i, i.e., the fraction of the estimated miss to be corrected ($0 \leq d \leq 1$)

x_i = the state of the system at time t_i, i.e.,

$$x_i = [m^*, \sigma, c]_{t_i}$$

$E[x_{i+1}]$ = the expected value of the state x_{i+1} which follows from making the decision d_i at the time t_i, starting in state x_i.

$E[\hat{p}(x)]_i$ = the expected value of the penalty function, starting in state x at t_i and employing an *optimal* policy to the final time t_f.

If the trajectory is divided into a sequence of decision times

$$(t_0, t_2, \cdots, t_i, \cdots, t_f)$$

where the option of making a correction is available, then[2]

$$E[\hat{p}(x)]_i = \underset{0 \leq d_i \leq 1}{\text{minimum}} \; E[\hat{p}(E[X_{i+1}])]_{i+1} \tag{24}$$

where $E[x_{i+1}]$ is computed from

$$E[m^*_{i+1}] = [1 - d_i]\, m^*_i \tag{25}$$

$$E[\sigma^2_{i+1}] = \alpha_{i+1} = \begin{cases} \left(\dfrac{\beta_i \gamma_i}{\beta_i + \gamma_i}\right) & \text{if } d_i > 0 \\[2ex] \left(\dfrac{\alpha_i \gamma_i}{\alpha_i + \gamma_i}\right) & \text{if } d_i = 0 \end{cases} \tag{26}$$

$$E(c_{i+1}) = c_{i+1} = c_i - \frac{d_i m^*_i}{\gamma_i} \tag{27}$$

where α_i, β_i, and γ_i are defined in part III. At the final time t_f,

$$E[p(x)]_f = \underset{0 \leq d_f \leq 1}{\text{minimum}} \left\{ [\alpha_f + (m^*_f)^2], \; [\beta_f + (1 - d_f)^2 (m^*_f)^2] \right\} \tag{28}$$

The process of generating the tabular function $E[p(x)]_i$ could present a difficult computational problem, but it will be shown below that d_i can be determined quite simply by considering only the instantaneous state of the system.

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[2] It is assumed that there is sufficient correction capability at t_i to perform a total correction, i.e., $d_i = 1$ is a legitimate case. Where this is not true is discussed in part VII.

VI. DETERMINATION OF THE OPTIMAL POLICY

In this part it will be shown that the optimal policy can be determined at any time t_i by a relatively simple computation if it is assumed that the effect of the proportional type of execution errors is sufficiently small. The result will be presented in the form of three theorems, leading to the construction of the optimal guidance policy discussed in the summary.

Definition 1. A total correction policy assumes that at each decision time t_i either no correction ($d_i = 0$) or a total correction ($d_i = 1$) is to be accomplished.

Definition 2. A two-correction policy assumes (at each decision-time t_i) that at most two corrections can be made: one at the final decision time t_f and another at some time $t_j < t_f$.

Theorem 1. The optimal two-correction policy is a total correction policy if the effect of the proportional type of execution error is sufficiently small.

Proof. If the optimal two-correction policy requires a correction at t_i it follows from Definition 2 that the resultant penalty function is similar to Eq. 22. Thus,

$$\frac{\partial\, \mathrm{E}\,[p(x)]_i}{\partial d_i} = \left(\frac{\partial \omega_i}{\partial d_i}\right) + \left(\frac{\partial \Psi_i}{\partial d_i}\right)\left(g + \mathrm{E}\,[a^2] + \Psi_i\, h\left(\frac{\partial \lambda}{\partial d_i}\right)\right) \tag{29}$$

where

$$h = \left(\frac{dg}{d\lambda_i}\right) = 4 \int_{\lambda_i}^{\infty} f(z)\,(z - \lambda_i)\,dz \tag{30}$$

Now, if the proportional execution error is sufficiently small it follows that

$$\mathrm{sign}\left\{\frac{\partial\, \mathrm{E}\,[p(x)]_i}{\partial d_i}\right\} = \mathrm{sign}\left\{\Psi_i\, h\left(\frac{\partial \lambda_i}{\partial d_i}\right)\right\} = (-) \tag{31}$$

since

$$\mathrm{sign}\left(\frac{\partial \lambda_i}{\partial d_i}\right) = \mathrm{sign}\left\{\left|\frac{m_i^*}{(\Psi_i)^{1/2}}\right|\left(1 - \frac{\tau_f}{\tau_i}\right)\right\} = (+) \tag{32}$$

and sign $h = (-)$. Thus, if a correction is made at t_i the optimal value of d_i is the maximum, i.e., $d_i = 1$.

Theorem 2. The optimal multiple-correction policy is the optimal two-correction policy if the effect of the proportional type of execution error is sufficiently small.

Proof. Suppose that for some $x\ (t_i)$ the optimal multiple-correction policy determined at t_i dictates a correction at t_{i+k}, t_{i+k+j}, and t_f. From theorem 2 it follows that there must be a total correction at time t_{i+k+j} and, by similar reasoning, the same conclusion applies to time t_{i+k}. But the expected value of the estimate at t_{i+k+j} would then be zero which implies no correction at that time. This contradiction immediately extends to the n-correction case, which establishes the theorem.

Definition 3. The switching function is

$$s_i(x) = E\,[p(x)]_i - \mathrm{E}\,[p_{i+1}(x)]_i \tag{33}$$

where $\mathrm{E}\,[p(x)]_i$ is the expected value of $p(x)$ evaluated at t_i, given that there is a total correction at t_i, and $E\,[p_{i+1}(x)]_i$ is the expected value of $p(x)$ evaluated at t_i, given that there is a total correction at t_{i+1} but none at t_i. The state x is extrapolated from t_i to t_{i+1}, as described in part V.

Theorem 3. Suppose that at least one correction is to be made between t_i and t_f. Then the optimal multiple-correction policy consists of setting $d_i = 1$ for $s_i \leq 0$, and setting $d_i = 0$ for $s_i > 0$.

Proof. From theorems 1 and 2 it follows that a total correction is to be performed at t_i or t_{i+1} or at some later time. A necessary condition to attain a stationary value (maximum or minimum) of the expected penalty by applying a correction at t_i is that $s_i = 0$. Since a minimum is sought, the theorem follows if there exists only one minimum point, which will be assumed. This completes theorem 3 and establishes the optimal guidance policy discussed in the summary.

VII. THE DEPLETION MODE OF OPERATION

It is assumed above that at each decision time t_i there is sufficient propulsion capability to perform a total correction, and that an unlimited number of corrections can be made during the remainder of the mission. Neither of these assumptions is realistic, however, for it is possible to deplete the propellant reserves, and engineering constraints may limit the total number of corrections.

Definition 4. The depletion mode of operation occurs at t_i when

$$n < 2 \text{ and/or } c < \frac{m_i^*}{\tau_i}$$

where n is the total number of corrections that can be performed at the decision times $(t_i, t_{i+1} \cdots t_f)$.

Without further justification, the following intuitively obvious policy will be adopted:

The Depletion Policy. The optimal policy for the depletion mode of operation is to correct as much of the error as possible at t_i when $\hat{s}_i \leq 0$, where

$$\hat{s}_i = \left[\beta_i + (r_i)^2\right] - \left[\beta_{i+1} + (r_{i+1})^2\right]$$

and

$$r_i = \begin{cases} 0 & \text{if } c\,\tau_i \geq m^* \\ m^* - c\,\tau_i & \text{if } c\,\tau_i < m^* \end{cases}$$

c = correction capability at t_i

m^* = estimate of target error at t_i

β_i = uncertainty resulting from orbit determination and execution errors, assuming a correction only at t_i

The quantities r_{i+1} and β_{i+1} are similarly defined. Notice that n effectively becomes a new state-variable.

VIII. EXTENSION TO MULTIPLE DIMENSIONS

The analysis has, thus far, considered only the simple case where one miss-component need be dealt with, but, in general, it is necessary to estimate all random variables that affect the observed data in order to obtain a minimum variance estimate of the orbit parameters (Ref. 9). Thus, all position and velocity components must be estimated, as well as unknown biases in the measuring devices and errors in the physical constants which describe the mathematical model. It is also necessary to consider more than one miss-component in order to compute the probability of impacting the target area. This general case can be treated in the manner presented above, however, by interpreting the variances associated with the idealized problem as being traces of certain combinations of covariance matrices. In this way a corresponding one-dimensional problem is constructed. The justification for this approach will not be rigorously established, but it will be shown that the penalty function determined in this way actually bounds the true result.

If Γ_i is the covariance matrix describing the error in the total estimate vector at t_i, and if there are no corrections in the interval (t_i, t_{i+k}), the covariance of the error in the total estimate vector at t_{i+k} is (Ref. 8)

$$\Gamma_{i+k} = \left[\Gamma_i^{-1} + \sum_{j=i}^{k-1} J_j\right]^{-1} \tag{34}$$

where J_j is the generalized inverse (normal matrix) of the covariance matrix describing the error in estimate due to observations gathered in the interval (t_j, t_{j+1}). If a correction is accomplished at t_i the covariance matrix Γ_i is replaced with

$$\Lambda_i = \Gamma_i + E\left[\delta\mathbf{v}_i\,\delta\mathbf{v}_i^T\right] \tag{35}$$

where E $[\delta \mathbf{v}_i \; \delta \mathbf{v}_i^T]$ is the covariance added by the random velocity execution errors. Let $\mathbf{m}$ be the n-dimensional target error vector that is to be nulled, and define the following relationships

$$m_i^* = |\mathbf{m}_i^*| \tag{36}$$

$$v_i = \left| \left[\frac{\partial \mathbf{m}}{\partial \mathbf{v}_i} \right]^{-1} m_i^* \right| \tag{37}$$

$$\alpha_i = \underset{\mathbf{m}\ \text{components}}{\text{trace}} [\Gamma_i] \tag{38}$$

$$\beta_i = \underset{\mathbf{m}\ \text{components}}{\text{trace}} [\Lambda_i] \tag{39}$$

$$\omega_i = \underset{\mathbf{m}\ \text{components}}{\text{trace}} \left[\Lambda_i^{-1} + \sum_{j=i}^{i_f - 1} J_j \right]^{-1} \tag{40}$$

$$\Psi_i = \beta_i - \omega_i \tag{41}$$

The quantity E $[r^2]_i$ can be determined for the general case by evaluating a multiple integral. If the variances of the individual components of the estimate of the target error at t_f are all equal, it follows that

$$[r^2]_i = \left(\frac{2}{\pi}\right)^k (\Psi_i) \int_{\lambda_i}^{\infty} (z - \lambda_i)^2 (z^{n-1}) \, exp\left(\frac{-z^2}{2}\right) dz$$

where $n = 1, 2$, or 3 is the dimension of $\mathbf{m}$, and $k = \left|\frac{n-2}{2}\right|$. With these relationships established, the analysis proceeds as in the one-dimensional case.

A corresponding one-dimensional problem has been constructed by the above process, but its physical interpretation is not obvious. It can be shown, however, that if a small residual target error estimate $\mathbf{r}$ exists at t_f, and if the probability of attaining a one-dimensional target error within some specified limits is computed by assuming β_f is the actual variance of the associated one-dimensional-probability-density-function, then the value so obtained is always less than or equal to the probability of impacting the corresponding multi-dimensional region (Fig. 5). If, following the discussion in part IV, the optimization problem discussed in this report is interpreted as being the maximization of the probability of impacting some given target region, it follows that the penalty function associated with the idealized problem will bound the value obtained for the general case, which it pretends to represent. The conclusion is that the resultant guidance policy will perform at least as well when applied to the general case as it does in the simplified problem.

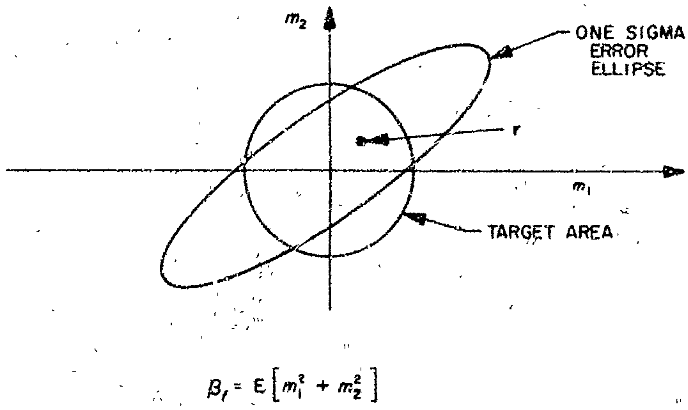

Fig. 5. The two-dimensional target error

IX. APPLICATION OF THE GUIDANCE POLICY

The guidance policy developed above was applied to a numerical example in order to demonstrate its effectiveness. The mathematical model describing the system was given in part III, with the parameters defining the problem chosen so as to reasonably represent a typical Mars-approach guidance situation (Table 1). For example, the final time t_f of approximately 15 hr before impact might correspond to the splitting off of an entry capsule from the spacecraft. To avoid a Monte-Carlo simulation, a "k-sigma" case was constructed by assuming that the estimated target error at each time t_i was k times the standard deviation of the estimate (over the ensemble of all experiments). The switching function was computed by using this simulated value. Thus, initially the estimate would be zero (at $t_0 = 0$); as tracking data was gathered it would build asymptotically toward $k\ (\alpha_0)^{1/2}$, be corrected to zero at the first correction time, and the process then re-initiated with β_1 replacing α_0. It was assumed that the correction capability initially was 20 m/sec; this number being chosen to adequately handle the "3-sigma case." The computer program developed to do this analysis is described in Tables 2 and 3. For these computations $g(\lambda)$ was approximated by

$$g(\lambda) = exp - (q_1\lambda + q_2\lambda^2) \tag{42}$$

Table 1. Parameter values defining the idealized approach guidance problem

Symbol	Description	Value
T	Time from start to impact.	10^6 sec
V	Spacecraft speed	5 km/sec
$(\alpha_0)^{1/2}$	Standard deviation of a priori orbit determination error.	10^3 km
Δt	Interval between decision times and tracking (star) observations.	$5\,(10^5)$ sec
σ_θ	Standard deviation of noise on the tracking (star) observations.	10^{-3} rad
τ_f	Time from impact at the final correction opportunity.	55×10^3 sec
σ_a	Standard deviation of the proportional execution errors.	0.01
σ_b	Standard deviation of the nonproportional execution errors.	0.1 m/sec

Table 2. Nomenclature for idealized guidance problem[a]

c	correction capability (m/sec)
$E\,[a^2]$	variance of proportional type of execution error (dimensionless)
$E\,[b^2]$	variance of nonproportional type of execution error (m/sec)^2
k	the *sigma level* of the simulated estimate
m_i^*	minimum-variance-estimate of target error predicted at t_i (kms)
q_1, q_2	constants in the exponential approximation of $g\,(\lambda)$
t_f	final decision time (sec)
t_i	time at i^{th} decision point (sec)
V	(constant) speed of the spacecraft toward the target (m/sec)
v_i	velocity impulse correction perpendicular to direction of motion applied at t_i (m/sec)
α_i	variance of error in the estimate m_i^*, assuming no correction at t_i (km)^2
β_i	variance of error in the estimate m_i^*, given a correction at t_i (km)^2
γ_i	variance of error in the estimate m_{i+1}^*, considering only orbit determination data in the interval $\{t_i, t_{i+1}\}$ (km)^2
Δt	time between decision points (sec)
η_f	same as η_i, with t_f replacing t_i
η_i	variance of error in the estimate m_i^*, considering only orbit determination data in the interval $\{t_0, t_i\}$ (km)^2
ρ_i	variance of error in the estimate m_f^*, considering only orbit determination data in the interval $\{t_i, t_f\}$ (km)^2
σ_θ	standard deviation of uncorrelated noise on each angular observation of the (dimensionless) star angle (Fig. 1)
τ_i	time-to-go to closest approach, evaluated at t_i (secs)
Ψ_i	variance of the estimate m_f^*, assuming a correction only at t_i and considering all orbit determination data (km)^2
ω_i	variance of the error in the estimate m_f^*, assuming a correction only at t_i and considering all orbit determination data (km)^2

[a] See Table 3 for equations describing the quantities defined here.

Table 3. The guidance policy logic

Input: $\tau_0, V, \Delta t, \alpha_0, \sigma_\theta, k, c, E\,[a^2], E\,[b^2], \eta_f, \tau_f, q_1, q_2$ Enter at time t_i, where $t_0 < t_i < t_f$. Let $\tau_i = \tau_0 - i\Delta t$. Proceed as follows:	**B. Computation of simulated estimate**
A. Orbit determination computations	$m_i^* = k\,(\alpha_0 - \alpha_i)^{1/2}$
$\gamma_{i-1} = (\sigma_\theta V \tau_{i-1})^2$ $\alpha_i = (\alpha_{i-1})(\gamma_{i-1})(\alpha_{i-1} + \gamma_{i-1})^{-1}$ $\eta_i^{-1} = \sum_{j=0}^{i-1} \gamma_j^{-1}$ $\rho_i = (\eta_f\,\eta_i)(\eta_i - \eta_f)^{-1}$	**C. Test for propellant depletion** $\Delta_i = c\,\tau_i - m_i^*$ If $\Delta_i \leq 0$ Go to propellant depletion mode of operation (part VII). If $\Delta_i > 0$ Continue

Table 3. Cont'd

D. Penalty for correction at t_i

$$\beta_i = \alpha_i + (m_i^*)^2 E[a^2] + \tau_i^2 E[b^2]$$

$$\omega_i = (\beta_i \rho_i)(\beta_i + \rho_i)^{-1}$$

$$\Psi_i = \beta_i - \omega_i$$

$$\lambda_i = [c - (m_i^*)(\tau_i)^{-1}](\tau_f)(\Psi_i)^{-1/2}$$

$$g_i = \exp - (q_1 \lambda_i + q_2 \lambda_i^2)$$

$$E[p]_i = \omega_i + \Psi_i g_i + \Psi_i E[a^2] + \tau_f^2 E[b^2]$$

E. Penalty for no correction until t_f

$$\omega_{i0} = (\alpha_i \rho_i)(\alpha_i + \rho_i)^{-1}$$

$$\Psi_{i0} = \alpha_i - \omega_{i0}$$

$$\lambda_{i0} = (c\,\tau_f - m_i^*)(\Psi_{i0})^{-1/2}$$

$$g_{i0} = \exp - (q_1 \lambda_i + q_2 \lambda_i^2)$$

$$E[p_0]_i = \omega_{i0} + \Psi_{i0} g_{i0} + \Psi_{i0} E[a^2] + \tau_f^2 E[b^2]$$

F. Test for no correction at t_i

$E[p]_i - E[p_0]_i \geq 0$ Make no correction. Go to time t_{i+1}. Restart computations

$E[p]_i - E[p_0]_i < 0$ or $\lambda_{i0} \leq 0$ Continue

G. Predicted penalty for correction at t_{i+1}

$$\gamma_i = (\sigma_\theta V \tau_i)^2$$

$$\alpha_{i+1} = (\alpha_i \gamma_i)(\alpha_i + \gamma_i)^{-1}$$

$$\eta_{i+1}^{-1} = \eta_i^{-1} + \gamma_i^{-1}$$

$$\rho_{i+1} = (\eta_f \eta_{i+1})(\eta_{i+1} - \eta_f)^{-1}$$

$$\beta_{i+1} = \alpha_{i+1} + (m_i^*)^2 E[a^2] + (\tau_{i+1}^2) E[b^2]$$

$$\omega_{i+1} = (\beta_{i+1} \rho_{i+1})(\beta_{i+1} + \rho_{i+1})^{-1}$$

$$\Psi_{i+1} = \beta_{i+1} - \omega_{i+1}$$

$$\lambda_{i+1} = [c - (m_i^*)(\tau_{i+1})^{-1}](\tau_f)(\Psi_{i+1})^{-1/2}$$

$$g_{i+1} = \exp - (q_1 \lambda_{i+1} + q_2 \lambda_{i+1}^2)$$

$$E[p_{i+1}]_i = \omega_{i+1} + \Psi_{i+1} g_{i+1} + \Psi_{i+1} E[a^2] + \tau_f^2 E[b^2]$$

H. Test for correction at t_i

$E[p]_i - E[p_{i+1}]_i > 0$ Make no correction. Go to time t_{i+1}. Restart computations.

$E[p]_i - E[p_{i+1}]_i \leq 0$ Continue (make correction)

I. Effect of correction at t_i

$$\nu_i = (m_i^*)(\tau_i)^{-1}$$

$$c = c - \nu_i$$

$$\alpha_i = \beta_i$$

Go to time t_{i+1}. Restart computations

where $q_1 = 1.5641$ and $q_2 = 0.36336$. The orbit determination statistics, assuming no corrections, are described in Fig. 6. The results for the 0.1, 1, 2, and 3-sigma cases are presented in Table 4 and Fig. 7 thru 10.

Table 4. Summary of results[a]

Sigma level	Correction number	Time-to-go at correction (sec × 10⁻³)	Correction applied (m/sec)	Total correction applied (m/sec)	Final rms error $(p)^{1/2}$ km
.1	1	55	1.82	1.82	87.20
1	1	390	2.47		
	2	55	4.76	7.23	87.32
2	1	335	5.79		
	2	55	8.43	14.22	87.[illegible]
3	1	315	[illegible]		
	2	150	3.71		
	3	55	6.66	19.63	87.63

[a]Total correction capability used constrained to be less than 20 m/sec.

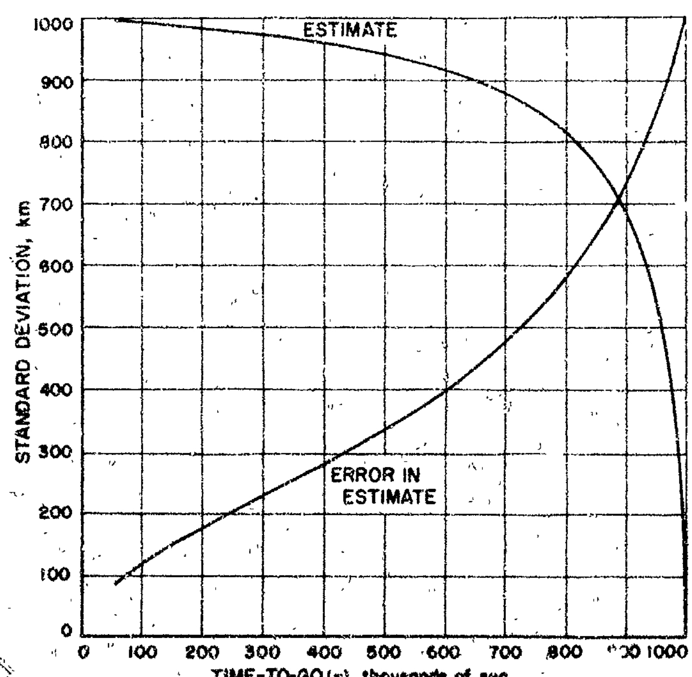

Fig. 6. Standard deviation of estimate and error in estimate vs time-to-go, assuming no corrections

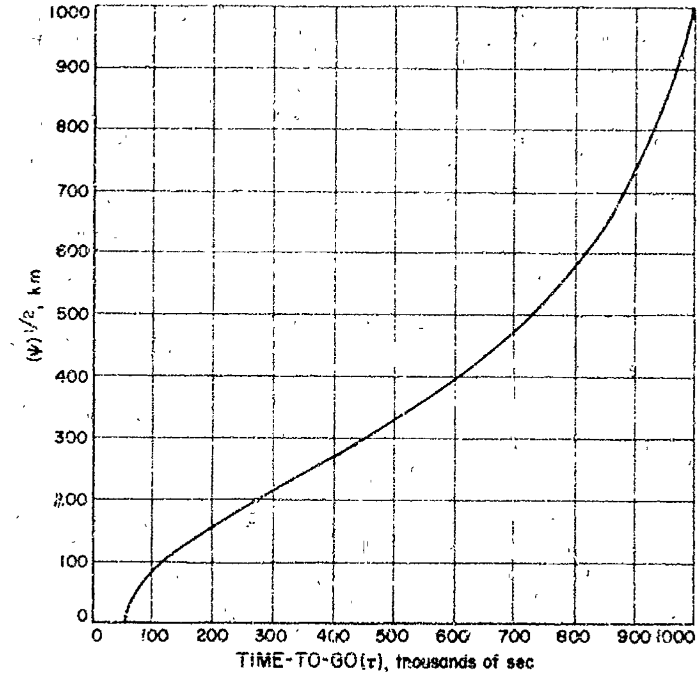

Fig. 7. The standard deviation of the estimate at t_f assuming a correction at τ, for 1, 2, and 3 signal levels

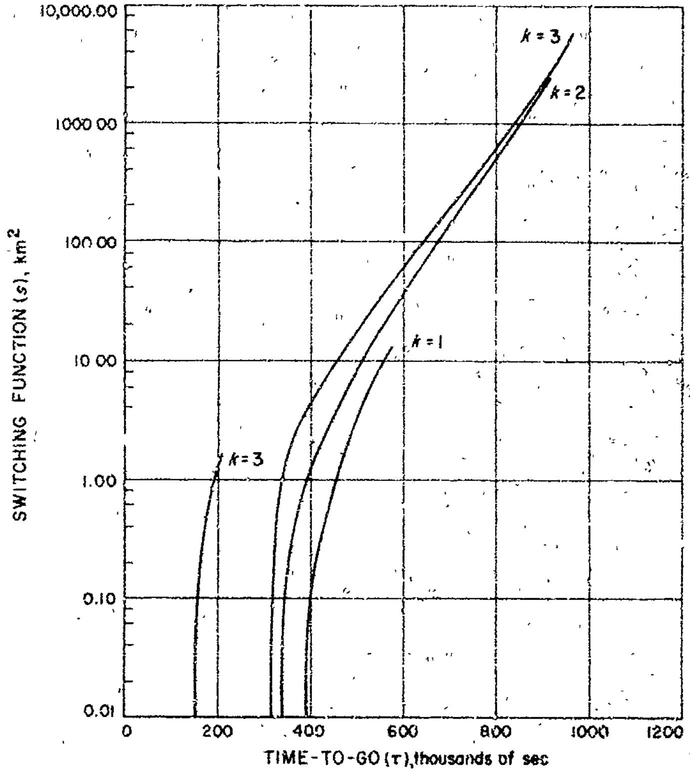

Fig. 9. The switching function vs time-to-go for various sigma levels

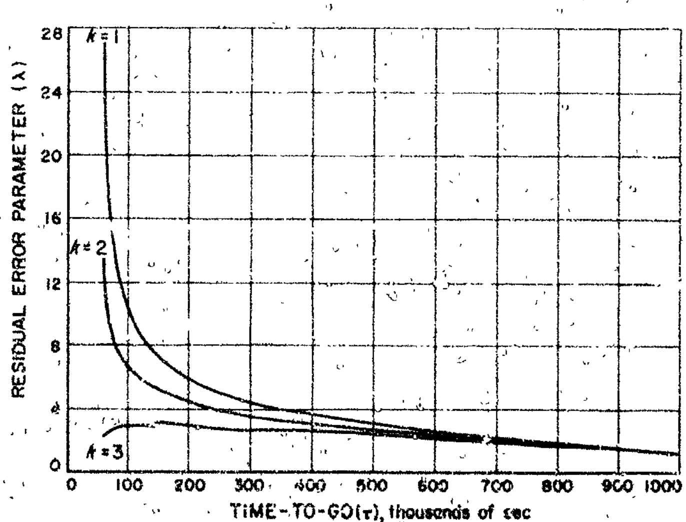

Fig. 8. The residual error parameter vs time-to-go

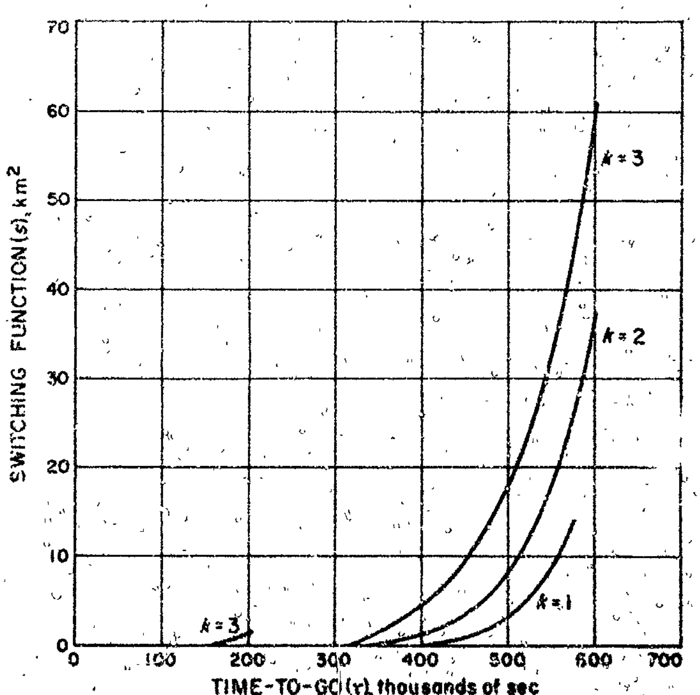

Fig. 10. A magnified view of the switching function for various sigma levels

A. DISCUSSION

An adaptive guidance correction policy has been developed which minimizes the expected value of the squared target error, subject to the constraint that the total propellant expenditure be less than some specified amount. This criterion is a good one for the case where the mission terminates at the final time, for then the highest degree of accuracy is usually desired, and there is no particular advantage in finishing with propellant left over. The scheme is well adapted for use in the real-time operational situation. The analysis has been carried out only for the idealized case, but an extension to the general case has been outlined.

The computational difficulties inherent in the dynamic programming formulation of the problem have been eliminated by developing the optimal policy in terms of the instantaneous state of the system. In order to accomplish this simplification it was assumed that the effect of the proportional type of execution error is negligible, which is the case when the corrections to be accomplished are small. It should be noted that the *result* of following this optimal policy is not directly available from the analysis, and must be obtained by a computer simulation of the mission, with Monte-Carlo selection of all random inputs which affect the trajectory of the spacecraft. This is no real limitation, however, for such simulations are usually performed in order to check the guidance logic.

One of the prime advantages of the guidance policy discussed in this report is that it tends to require a minimum number of corrections, usually two. This is important because each correction degrades the reliability of the spacecraft, disturbing it from the normal cruise mode and subjecting it to potential failures in the subsystem which commands the correction. Further studies are planned to continue the evaluation of this guidance technique.

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